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Liquid impact, kinetic energy loss and compressibility: Lagrangian, Eulerian and acoustic viewpoints

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Abstract. From Lagrange's equations of incompressible fluid motion a model is derived for the collision between a liquid mass and a solid surface. The classical idea of pressure impulse, P, is re-expressed as a quantity following the fluid-particle motion. It is shown that within this formulation $P = 0$ is the exact free-surface boundary condition and the domain of definition of P is unambiguously time-independent. Some of the total kinetic energy of the fluid is lost during impact and this is associated with the usual choice of boundary condition for inelastic impact. With elastic impact, in which the fluid rebounds from the solid target, there is no kinetic energy loss. Some simple potentials are used to express P for incompressible fluid impacts, which have non-singular velocity fields: (i) in an acute wedge; (ii) in a cylindrical container; and (iii) in an idealised sea-wave impact. In the last the impact of a triangular fluid domain, T, illustrates kinetic energy loss from an impacting sea wave. Impact is also investigated for the collision of T with a movable solid block. The subsequent displacement of the block, with friction, is also calculated. Lastly a solution is obtained within T composed of a compressible fluid impacting a rigid wall. Standing compression-waves store within T some of the kinetic energy lost from the incident wave water.

Key words: compressibility, energy, impact, impulsive force, water wave.

1. Introduction

The collision between a liquid mass and a solid surface can be modelled by methods dating from Lamb ([1, Section 11]), who treated the impulsive motion of a liquid impelled from rest. The application of the theory to moving bodies of water meeting solid surfaces at rest has been developed by Cooker and Peregrine ([2],[3]), Chan [4] and Cooker and Vanden-Broeck [5], to encompass the impact of a sea wave against a coastal structure. Korobkin [6] has modelled the influence of fluid compressibility. Korobkin and Pukhnachov [7], and Zhang, Yue and Tanizawa [8] have described the effect of the elastic deformation of the target. But in most studies the solid surface struck by the fluid is rigid. Korobkin and Wu [9] have used Lagrangian variables to calculate the free-surface displacement due to a suddenly moved floating cylinder.

The theory used in the present work relies upon the idea of pressure impulse, P, a function of position within the fluid domain at the instant of impact. The pressure impulse is classically defined to be the time-integral of the pressure over the brief duration, Δt, during which the impact occurs. From pressure records at fixed locations, experiments show that the pressure rises rapidly and falls in a characteristic spike. We identify the width of this spike in pressure with Δt. The factors which govern the short time-scale Δt are not well-understood, but for an incompressible fluid it may be related to the time it takes for the moving contact line to travel across that part of the boundary which can be identified as the zone of impact. Typical laboratory-scale measurements show Δt is very brief compared with the usual time scales associated with water wave propagation: the pressure rises and falls between 0·5 and 10 milliseconds. And at full scale Δt is longer: up to 100 milliseconds. These measured impact times are much briefer than the corresponding period of the incident waves (between 1 and 10 seconds). Further, for a breaking wave of height L and incident water speed U_0, the time scale L/U_0 is 0·1 seconds for small scale waves in the laboratory and about 1 second in the field. We conclude from the reported measurements that Δt is much shorter than L/U_0.

Bagnold [10] discussed the concept of pressure impulse for fixed points on a vertical wall, at which he made pressure measurements, of waves striking the wall with height approximately 10cm. Here we emphasise the treatment of pressure impulse as a field whose gradient is crucial to understanding the changes which occur to the velocity field during impact.

If U_0 is a typical flow speed, and if L is a length scale of the impacting fluid region, then we can define primed, dimensionless variables as follows: velocity $\mathbf{u}' = \mathbf{u}/U_0$, coordinates $\mathbf{x}' = \mathbf{x}/L$, time $t' = t/\Delta t$, and pressure $p' = p/p_0$, where we defer the definition of the constant p_0. The time scale L/U_0 is associated with temporal changes in the incident flow *before* impact; L/U_0 is much longer than the time Δt during which the impact occurs. Since we are focussing on the unsteadiness *during* impact it is sensible to make time t dimensionless with respect to Δt and not L/U_0. Consequently, in dimensionless form, Euler's equations are

$$\mathbf{u}'_{t'} + \epsilon(\mathbf{u}'.\nabla')\mathbf{u}' = -\frac{p_0\Delta t}{\rho L U_0}\nabla' p' - \frac{g\Delta t}{U_0}\mathbf{k}. \tag{1}$$

In Equation (1) the dimensionless quantity $\epsilon = U_0\Delta t/L$ can be used to indicate the violence of impact. The magnitude of $1/\epsilon = p_0/(\rho {U_0}^2)$, hence the smaller the value of ϵ the greater the relative impact pressure scale p_0. When ϵ is much less than unity, provided $|(\mathbf{u}'.\nabla')\mathbf{u}'|$ is not large, Equation (1) shows that one may neglect the nonlinear terms of Euler's equations. Further, the acceleration due to gravity, $-g\mathbf{k}$, has a modulus which in practice is small compared with that of the fluid acceleration during impact (of magnitude $U_0/\Delta t$). It is therefore true that $|g\Delta t/U_0| \ll 1$, and the dimensionless gravitational term in Equation (1) is negligible compared with the inertia of the fluid.

In order to obtain an $O(1)$ coefficient for the pressure-gradient term (and so provide some balance of terms in (1)) it is necessary to take $p_0 = \rho L U_0/\Delta t$. A consequence of these assumptions is that the expected pressures of impact are $p_0 \gg \rho g L$, hence Froude-Law scaling is inappropriate. Also the smaller the impact time Δt the greater the peak pressure scale p_0.

Neglecting the second and fourth terms, Equation (1) is integrated with respect to time over the brief duration, Δt, of the impact. For an incompressible fluid of constant density this simplifies to the following dimensional form:

$$\mathbf{V_2} - \mathbf{V_1} = -\rho^{-1}\nabla P, \tag{2}$$

where P is the pressure impulse, defined above and $\mathbf{V}_1$ and $\mathbf{V}_2$ denote the fluid velocity just before, and just after impact, respectively. The subscripts 1 and 2 on a quantity denote before and after impact, respectively, throughout. In typical boundary-value-problems one usually knows $\mathbf{V}_1$ and wishes to find $\mathbf{V}_2$.

If the fluid is incompressible then the divergence of Equation (2) shows that P satisfies Laplace's equation:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = 0. \tag{3}$$

Next consider fluid colliding with a solid, rigid, impermeable surface which has a unit normal $\mathbf{n}$, pointing into the solid, away from the fluid. Let V_{1n} denote the velocity component in the direction $\mathbf{n}$. If the fluid remains in contact with the boundary after impact then the scalar product of $\mathbf{n}$ with Equation (2) gives the boundary condition

$$\frac{\partial P}{\partial n} = \rho V_{1n}. \tag{4}$$

This corresponds to the familiar *inelastic* impact on a rigid surface in classical particle mechanics. By contrast, for an *elastic* impact $V_{2n} = -V_{1n}$ and the RHS of (4) must be doubled. Such a boundary condition was first used by Wood [11] and Wood, Peregrine and Bruce [12]. If the solid target is movable then further information about the flow before and after impact must be obtained from an equation of motion of the target. We discuss this further in Section 5. On those parts of an impermeable boundary where no impact occurs $V_{1n} = 0$ and the boundary condition is $\partial P/\partial n = 0$.

Lastly, on a free surface, the pressure is zero (reference) for all time, so it is customary to take the boundary condition as

$$P = 0. \tag{5}$$

It was pointed out by Cooker and Vanden-Broeck [5], that the free-surface boundary condition (5) is problematic. For some geometries, during impact the waterline can move at such high speed that one might doubt that ϵ remains small. Further, a fixed geometric point, with position vector $\mathbf{r}$, which lies on the free surface position at the initial instant of impact, will be left behind as the free surface moves. Either the point at $\mathbf{r}$ is left high-and-dry or it becomes submerged. At $\mathbf{r}$ the pressure may deviate somewhat from zero. So is $P = 0$ the appropriate free-surface boundary condition?

We answer this question in the affirmative in the next section by deriving a model from Lagrange's equations of motion. We go on in Section 3 to discuss energy conservation. In Section 4 some examples of pressure impulse are presented, which although mathematically simple reveal much about the behaviour of impacting fluid flows, especially near waterlines. An exact solution in a right-angled, isosceles triangle, T, is a convenient non-singular expression for P with which to illustrate energy conservation. In Section 5, T collides with either a fixed or a movable solid mass. In Section 6 we find an exact solution of the acoustic equations suitable for when the fluid in T is slightly compressible. Conclusions are drawn in Section 7.

2. Analysis from Lagrange's equations

Let $\mathbf{X}(\mathbf{x}, t)$ denote the position vector of F a fluid particle. At the start of impact, $\mathbf{X} = \mathbf{x}$, so that we can interpret $\mathbf{x}$ as an unchanging label for F. The components of $\mathbf{x}$ and t are independent variables. The velocity of F at any time is

$$\mathbf{V}(\mathbf{x}, t) = \frac{\partial \mathbf{X}}{\partial t}. \tag{6}$$

We will now describe an impact in which $\mathbf{V}$ changes rapidly from $\mathbf{V}_1(\mathbf{x})$ to $\mathbf{V}_2(\mathbf{x})$ in the short time interval $[-\tau, \tau]$, where $\tau = \Delta t/2$. The constant $\tau > 0$ is assumed to be the same throughout the fluid domain; a reasonable assumption for a fluid in which pressure changes are communicated at a sound speed much greater than the fluid speed. Before impact ($t < -\tau$) the velocity field changes relatively slowly, on a time-scale T which is much longer than the impact time scale τ. Similarly, after impact ($t > \tau$) we suppose $\mathbf{V}_2$ changes slowly. However, during impact ($-\tau \leq t \leq \tau$) the fluid acceleration is much greater than at all other times. These considerations can be encapsulated in the following *ansatz* for the position vector $\mathbf{X}$ of the fluid particle F during impact:

$$\mathbf{X}(\mathbf{x}, t) = \mathbf{x} + h_1(t)\mathbf{V}_1(\mathbf{x}) + h_2(t)\mathbf{V}_2(\mathbf{x}), \tag{7}$$

where the functions h_1 and h_2 are arranged to ensure that $\partial\mathbf{X}/\partial t$ changes from $\mathbf{V}_1$ to $\mathbf{V}_2$ in the time interval $[-\tau, \tau]$. Now h_1 and h_2 must be smooth enough functions of time to ensure that the fluid velocity changes smoothly from V_1 to V_2. Both h_1 and h_2 must have (a) a magnitude of order τ and (b) continuous first derivatives of magnitude no greater than order (1). The violation of (b) would suggest that an impact was occurring on a yet shorter time scale than τ. In order that the theory can encompass flows for which there is a vanishingly small change in the velocity field ($\mathbf{V}_1 = \mathbf{V}_2 = \partial\mathbf{X}/\partial t$) we need $\dot{h}_1(t) + \dot{h}_2(t) = 1$ for all t. We do not have to specify the functions explicitly to reach the key conclusion below (Equation (17)), but for definiteness we provide an example of functions which will do

$$\begin{aligned} h_1(t) &= \tau + t && \text{for} \quad t :\leq -\tau, \\ h_1(t) &= \tau - (t-\tau)^2/(4\tau) && \text{for} \quad t : -\tau \leq t \leq \tau, \\ h_1(t) &= \tau && \text{for} \quad t : \tau \leq t, \end{aligned} \tag{8}$$

and

$$\begin{aligned} h_2(t) &= 0 && \text{for} \quad t : t \leq -\tau, \\ h_2(t) &= (t+\tau)^2/(4\tau) && \text{for} \quad t : -\tau \leq t \leq \tau, \\ h_2(t) &= t && \text{for} \quad t : \tau \leq t. \end{aligned} \tag{9}$$

The fluid acceleration is so much greater than the acceleration due to gravity, g, that we neglect gravity. Consequently Lagrange's equations of motion are:

$$X_{tt}X_x + Y_{tt}Y_x + Z_{tt}Z_x = -\rho^{-1}p_x, \tag{10}$$

$$X_{tt}X_y + Y_{tt}Y_y + Z_{tt}Z_y = -\rho^{-1}p_y, \tag{11}$$

$$X_{tt}X_z + Y_{tt}Y_z + Z_{tt}Z_z = -\rho^{-1}p_z, \tag{12}$$

where subscripts which are variables denote partial derivatives, $p(\mathbf{x}, t)$ is the pressure following a particle and ρ is the constant density. Since the treatment is similar for all three equations we consider Equation (10) alone. We integrate it with respect to time over the interval of impact $[-\tau, \tau]$. After integration by parts, we have

$$[X_tX_x + Y_tY_x + Z_tZ_x]_{-\tau}^{\tau} - \frac{1}{2}\frac{\partial}{\partial x}\int_{-\tau}^{\tau}({X_t}^2 + {Y_t}^2 + {Z_t}^2)\mathrm{d}t = -\rho^{-1}P_x, \tag{13}$$

where the pressure impulse P is defined by

$$P(\mathbf{x}) = \int_{-\tau}^{\tau} p(\mathbf{x}, t)\mathrm{d}t. \tag{14}$$

Formally this definition of pressure impulse, P, is a different quantity from that presented in the introduction, because here the integration follows a fluid particle, F, through both time and space. However, for the most violent impacts (in the limit as $\epsilon \to 0$), the two definitions coincide. From (14), if F lies on the free-surface then the pressure $p = 0$. Hence on the free surface the pressure impulse $P = 0$, exactly. This confirms Equation (5).

Substituting (7) in (13) gives

$$U_2 - U_1 - \frac{1}{2}\frac{\partial \mathbf{V}_1{}^2}{\partial x}\int_{-\tau}^{\tau}\dot{h}_1^2 \mathrm{d}t + \frac{1}{2}\frac{\partial \mathbf{V}_2{}^2}{\partial x}\left[\tau - \int_{-\tau}^{\tau}\dot{h}_2^2 \mathrm{d}t\right] - \frac{\partial \mathbf{V}_1\cdot\mathbf{V}_2}{\partial x}\int_{-\tau}^{\tau}\dot{h}_1\dot{h}_2 \mathrm{d}t + \\ + \tau \mathbf{V}_2\cdot\frac{\partial \mathbf{V}_1}{\partial x} = -\rho^{-1}P_x \tag{15}$$

where (U_1, V_1, W_1) are the components of $\mathbf{V}_1$ and (U_2, V_2, W_2) are those of $\mathbf{V}_2$.

In general, in (15), $\dot{h}_1$ and $\dot{h}_2$ are of order unity. Therefore all three integrals in Equation (15) are of order τ. (Hence, in the limit as $\tau \to 0$, we obtain the x-component of Equation (17) below.) For definiteness we calculate the integrals for the example functions (8, 9):

$$U_2 - U_1 + \frac{\tau}{6}(-4\mathbf{V}_1.\mathbf{V}_{1x} + 2\mathbf{V}_2.\mathbf{V}_{2x} - \mathbf{V}_1.\mathbf{V}_{2x} + 5\mathbf{V}_2.\mathbf{V}_{1x}) = -\rho^{-1}P_x. \tag{16}$$

The other two componenent equations are similar.

By using the scalings introduced for Equation (1) we can make (16) dimensionless, and the coefficient $(\tau/6)$ of the nonlinear terms is replaced by $\tau U_0/(6L) = \Delta t U_0/(12L) = \epsilon/12$. By letting $\tau \to 0$ we obtain the same limit as that presented in the introduction, by letting $\epsilon \to 0$; provided the nonlinear convective terms remain bounded in magnitude. Hence as $\epsilon \to 0$ we conclude the general result

$$\mathbf{V}_2 - \mathbf{V}_1 = -\rho^{-1}\nabla P, \tag{17}$$

where the gradient operator is with respect to either the Lagrangian point-label coordinates x, y, z or the identical (in this limit) Eulerian coordinates of Equation (2).

In Lagrangian coordinates, the equation of mass continuity for an incompressible fluid of constant density is

$$\frac{\partial(X, Y, Z)}{\partial(x, y, z)} = 1. \tag{18}$$

Substitution of the *ansatz* (7) gives the following approximation to Equation (18), in which the products $h_1{}^2, h_1h_2, h_2{}^2$ have been neglected compared with h_1 and h_2

$$h_1(U_{1x} + V_{1y} + W_{1z}) + h_2(U_{2x} + V_{2y} + W_{2z}) = 0. \tag{19}$$

By setting $t = 0$ then $t = \tau$ one deduces that $\nabla\cdot\mathbf{V}_1 = 0$ and $\nabla\cdot\mathbf{V}_2 = 0$. By taking the divergence of Equation (17) one finds that P satisfies Laplace's Equation (3). The boundary conditions on solid surfaces, in the impact zone and elsewhere, also coincide with those stated in the introduction.

The analysis agrees with the classical equations of pressure impulse theory, but with three differences of interpretation: (a) the pressure impulse P is now defined for each and every fluid particle, (b) the free-surface condition $P = 0$ is exact, and (c) the domain, D, in which a boundary-value problem for P is posed, is unambiguously time-independent. Here D is the domain of Lagrangian particle labels $\mathbf{x}$, and for all $\tau > 0$ it coincides with the region of fluid in physical space at the start of impact.

3. Energy conservation

Wu [13] highlighted a discrepancy between formulas which claim to account for the force on a rigid body during water entry. Scolan and Korobkin [14], and Fontaine, Molin and Cointe [17] have shown that if a rigid mass enters a half-space of initially still water, at constant velocity then half of the work done by the body on the fluid can be accounted for by the kinetic energy of high-speed jets. These splash-jets lie so close to the body that their resolution needs special asymptotic treatment in the examples they model. Korobkin [6, Equation 16], made a calculation which suggested that one quarter of the energy is radiated to infinity in the collision of a prismatic jet of fluid, at normal incidence, onto a plane, rigid wall. This result has been superseded by the work of Kovobkin and Pevegnine [18]. The proportion of energy radiated depends strongly on the variation of velocity during impact.

We now account for the energy of the flow before and after impact for our incompressible fluid model. During the short time of impact the liquid displacement is small because $\epsilon \ll 1$. Hence, in the following, any small changes in the gravitational potential energy are neglected.

Let KE_1 and KE_2 be the kinetic energies of the fluid domain before and after impact, respectively. Then the amount of kinetic energy lost during impact is

$$KE_1 - KE_2 = \int\int_D \frac{\rho}{2}\left(\mathbf{V}_1{}^2 - \mathbf{V}_2{}^2\right) \mathrm{d}x\mathrm{d}y, \tag{20}$$

where the integral is over the fluid domain D. Substituting for $\mathbf{V}_2$ from Equation (2), applying the divergence theorem to $\nabla\cdot(2\rho P\mathbf{V}_1 - P\nabla P)$, and using the boundary conditions, we arrive at an integral over only that part I of the solid boundary impacted by fluid:

$$KE_1 - KE_2 = \int_I P\left(V_{1n} - \frac{1}{2\rho}\frac{\partial P}{\partial n}\right) \mathrm{d}S. \tag{21}$$

where $\mathrm{d}S$ is an infinitessimal element of area of I and V_{1n} is the velocity component along the unit normal. This result is implicit in the work of Lamb ([1, articles 11, 44, 61]) and has been discussed by Wu [13], Scolan and Korobkin [14], Szymczak [15] and Rogers and Szymczak [16]. See also [2].

For impact in which the fluid remains in contact with the wall after impact, the boundary condition is $\partial P/\partial n = \rho V_{1n}$. But with this condition Equation (21) shows that there is a loss of (kinetic) energy from the fluid domain. Since the wall and bed are rigid, fixed and impermeable, the wave cannot lose energy through these surfaces during impact by, for example, doing mechanical work on the boundary. Since the fluid is incompressible the pressure in the fluid cannot do internal work, by for example self-heating. For a compressible fluid in a finite domain which has perfectly reflecting solid and free-surface boundaries, there is no chance for sound waves to radiate energy to infinity, although stationary pressure waves can store potential energy. Korobkin [6] makes clear that when acoustic waves are free to radiate energy to infinity they do so as a direct consequence of the constraint of fluid staying in contact with the wall during impact, despite the low pressures induced by reflections from the free surface.

The resolution of the paradox of energy loss in an incompressible fluid during impact lies in Equation (21) and our recognition that an extra physical condition is imposed in assuming that the fluid remains in contact with the wall after impact. If the fluid is in elastic impact with the wall, then the face of the wave bounces from the wall with a relative normal velocity component equal and opposite to its incident value, V_{1n}. The change in normal velocity at the wall is therefore $2V_{1n}$ and the wall boundary condition becomes $\partial P/\partial n = 2\rho V_{1n}$. From (21) it is seen that under these elastic-impact conditions the energy loss is zero. Elastic impact is the only impact-boundary condition which accords with energy conservation for an incompressible fluid impact.

4. Examples of pressure impulse for fluid impacts

4.1. Impact of a fluid wedge

When a sea wave overturns onto a beach or breaks against a seawall, a mass of water collides with a fixed impermeable surface. We investigate a two-dimensional flow corresponding to normal incidence of a sea wave onto a plane solid surface.

The fluid domain at the instant of impact is modelled as a wedge whose sides are the solid surface being struck and the free surface. Let the apex of the wedge lie at the origin and let the positive x-axis be the solid surface. The free surface is $y = x\tan\alpha$ where $\alpha : 0 < \alpha \leq \pi$ is the angle in the wedge immediately before impact. The fluid descends onto the x-axis with a normal velocity component $V_{1n} = V_0$ where the constant $V_0 > 0$. Hence the boundary condition is $\partial P/\partial y = -\rho V_0$ on $y = 0$. Now $P = 0$ on $y - x\tan\alpha = 0$, so we can guess the exact solution to Equation (3)

$$P(x, y) = \rho V_0(x\tan\alpha - y). \tag{22}$$

After impact the waterline, at O, moves with a velocity given by Equation (2) as $\mathbf{V}_2 = -\mathbf{i}\rho^{-1}P_x$. The waterline travels at speed $V_0\tan\alpha$ along the negative x-axis. The steeper the wave impact the faster the waterline travels.

The expression (22) gives $P \geq 0$ for acute wedge angles $\alpha : 0 < \alpha < \pi$. The isopotential curves of $P(x, y)$ are straight lines parallel to the free surface. However, for $\alpha : \pi/2 < \alpha < \pi$ the pressure impulse (22) is negative. Since P is the integral of the positive quantity pressure we expect $P < 0$ to be unphysical. A further observation of (22) is that P is singular for $\alpha = \pi/2$. Clearly, a different solution is needed for obtuse wedge angles.

Okamura [19] has calculated the pressure impulse for a wedge-shaped wave impacting the x-axis on the interval $x : 0 \leq x \leq 1$; on the remainder of the x-axis the fluid stays in contact with the solid boundary. Okamura also imposed a far-field boundary condition P tends to 0. He found that for $\alpha : \pi/2 \leq \alpha < \pi$ the speed of the waterline along the x-axis is singular at $x = 0$. This may indicate the cirumstances in which a high-speed jet or tongue of water can be projected up a beach by a wave impact.

4.2. Axisymmetric impact

Milne-Thomson [20, Examples XVII, Q44] leaves as an exercise the axisymmetric impact of a circular cone whose base collides with an impermeable solid surface. A modification of his potential can describe an impact on the interior curved walls of a circular cylinder; see Figure 1. Such an impact might occur during the shaking of a container of water, fuel, or food-stuff. More complex interior and exterior shaking problems, in cylindrical geometry, are treated by Jacobsen [21].

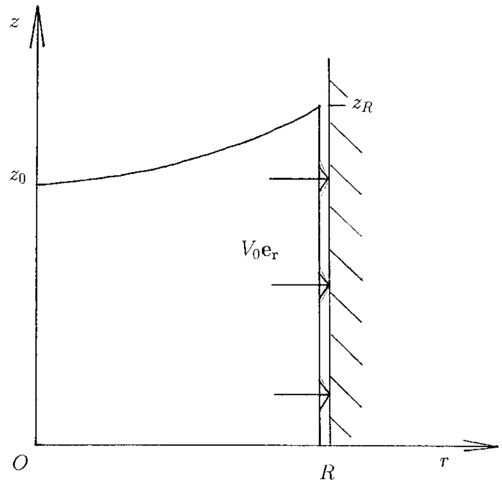

Figure 1. Cross-Section through the axis of an idealised fluid impact on the curved interior wall of a circular cylinder. The z-axis is the axis of symmetry.

Let r, z be cylindrical polar coordinates. The impermeable base of the cylinder lies at $z = 0$ and the curved wall lies at $r = R$, where $R > 0$ is the constant radius of the cylinder. The fluid lies inside the cylinder in part of the region $z \geq 0$. The fluid stays in contact with the bed during impact and it collides with the sidewall with a constant outward radial velocity $V_0\mathbf{e}_r$, where $\mathbf{e}_r$ is the unit radial vector. A positive potential P, which satisfies the boundary conditions on the wall and bed, is

$$P(r, z) = P_0 + \rho\frac{V_0}{2R}\left(r^2 - 2z^2\right), \tag{23}$$

where $P_0 \geq 0$ is a constant.

The free surface is any surface on which the boundary condition $P = 0$ is satisfied. If $P_0 = 0$ then the free surface is the cone $r = \sqrt{2}z$. More natural free-surface shapes can be made by choosing $P_0 \geq 0$. For example, if $P_0 = {z_0}^2\rho V_0/R$, then the free surface is a truncated hyperboloid which cuts the z-axis at $z = z_0$. The waterline on the cylinder wall lies at $z = z_R = z_0(1 + R^2/2z_0^2)^{1/2}$. By making $z_0 >> R$ the hyperboloidal free surface can be made as flat as desired.

One consequence of the impact on the walls is the significant impulsive force on the base of the cylinder. The impulse on the bed due to the fluid is $-2\pi\mathbf{k}\int_0^R P(r, 0)r\mathrm{d}r$, where $\mathbf{k}$ is the unit vector along the positive z-axis, which is normal to the bed. The impulse has magnitude

$$I = \pi\rho V_0 R^2 z_0\left(\frac{z_0}{R} + \frac{R}{4z_0}\right). \tag{24}$$

The waterline rushes up the cylinder wall after impact: the increase in vertical velocity due to the impact is W_R, where

$$W_R = -\frac{1}{\rho}\frac{\partial P}{\partial z}(R, z_R) = 2V_0\frac{z_0}{R}\left(1 + \frac{R^2}{2{z_0}^2}\right)^{1/2}. \tag{25}$$

We now present a numerical example. If the fluid depth at the z-axis is $z_0 = 2R$ then at the wall the depth is $z_R = 3z_0/(2\sqrt{2}) = 1{\cdot}06z_0$, hence the free surface is only slightly concave.

2.

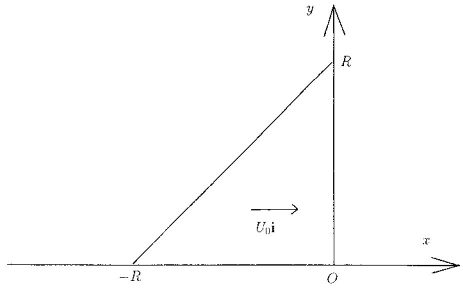

Figure 2. Notation for an idealised water-wave impact on a vertical wall. The wave meets the rigid wall with velocity $U_0\mathbf{i}$. The bed at $y = 0$ is impermeable.

3.

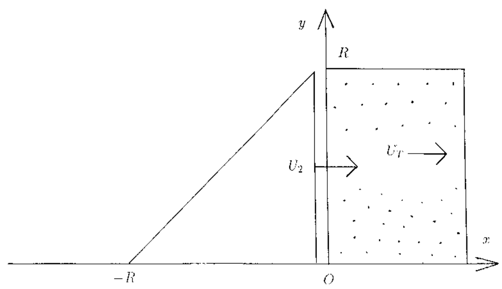

Figure 3. Wave impact on a movable block. For elastic impact the wave face separates from the wall after impact with a relative speed of separation $U_2 - U_t$ equal to U_0, the speed of impact. For inelastic impact the wave face remains in contact with the block, and they move forward together, with an initial post-impact speed of $U_2 = U_t$.

The waterline ascends with an increase in the vertical velocity of $W_R = 6V_0/\sqrt{2} = 4{\cdot}2V_0$; more than four times the speed of impact.

The impulse on the bed due to the fluid impacting the walls is

$$I = \frac{17}{8}\pi\rho V_0 R^2 z_0. \tag{26}$$

By way of comparison, I is more than twice $\pi\rho V_0 R^2 z_0$, which is the impulse on the bed due to the inelastic impact *onto the bed*, of a rigid body of mass $\pi\rho R^2 z_0$, travelling at speed V_0. The enhancement of impulse by a factor of more than two, on the container bed, is due to the close confinement of the fluid domain by impermeable walls. The increased impulse was illustrated by Cooker and Peregrine [3] in rectangular containers.

4.3. Two-dimensional impact: isosceles triangle

Cooker and Peregrine [2,3] solved some boundary-value problems suitable for modelling the impulsive flow generated by a sea wave impacting a coastal structure. See also Chan [4] for his theoretical and experimental study of wave impact in deep water.

A straight vertical seawall is approached by a train of long-crested plane water waves. It is recognised by coastal engineers that the highest forces are produced by waves which approach the structure with their crests parallel to the seawall. Immediately before impact a breaking wave can take a variety of shapes, but if it is of the plunging type then the wave may have a well defined steep forward face. The highest impact pressures occur when the wave face is parallel to the wall at the moment of impact. This tallies with the laboratory findings of Bagnold [10] and subsequent experimenters, *e.g.* Whillock [22]. Sea wave impact has recently been reviewed by Peregrine [23].

Cooker and Peregrine [3] consider a fluid domain which at the instant of impact has the form of a right-angled isosceles triangle. The mathematical expression for P takes its simplest form for an isosceles right-angled triangle, when the entire forward face of the wave simultaneously impacts the vertical wall.

Figure 2 shows the coordinates for a wave impacting a wall which is rigid (a movable wall is considered in the discussion of Section 5). The normal component of velocity of the vertical wave face immediately before impact is U_0, a positive constant whose value can be chosen in ways appropriate to the wave-breaking situation. In water of depth h photographic evidence suggests that U_0 is an order-one multiple of $\sqrt{gh}$. We assume that the wave face remains in contact with the wall after impact. Hence the boundary condition on $x = 0$ is $P_x = \rho U_0$. On $y = 0$ the bed is impermeable hence $P_y = 0$.

A suitable solution, which is positive in the triangular fluid domain, is

$$P(x, y) = \rho \frac{U_0}{2R} \left([x + R]^2 - y^2 \right). \tag{27}$$

The free-surface condition $P = 0$ is satisfied on the straight line $y = R + x$ for $x : -R \leq x \leq 0$. Immediately after impact the velocity is finite everywhere. In particular the waterline, at $x = 0$, $y = R$, ascends the wall with finite speed, U_0.

Below we discuss the energy before and after impact relative to the wall boundary condition. Instead of assuming the water stays in contact with the wall after impact, we follow Wood [11] and suppose that the fluid 'bounces back' elastically from it. By this we mean the (normal) fluid velocity component in the $\mathbf{i}$ direction, changes from U_0 before impact to $-U_0$ after impact. Then in Equation (27) we need only replace U_0 by $2U_0$ to obtain the appropriate solution. Consequently the velocity field after impact is more energetic. For example the top of the triangle after impact has double the vertical component of velocity: $2U_0$ and it acquires a horizontal velocity component U_0 of rebound from the wall.

5. Impact with a movable block

Suppose we abandon the fiction of a rigid wall and consider a block of finite mass. This has been analysed before by Goda [24, p.154] who treats a block restrained by elastic forces and friction. Under failure the block is unlikely to be elastically attached to its foundations or neighbouring blocks, which is why we model friction alone.

The block translates horizontally due to the impulse delivered by the wave impact. We neglect friction during the small time of impact. Whether elastic or inelastic impact is modelled, some kinetic energy is transferred from the wave to the block, and we calculate the loss for inelastic impact in Section 5.2. In Section 5.3 we estimate the displacement of the block after impact, when it moves under frictional resistance.

First an elastic impact is treated, then a more realistic inelastic impact is compared.

5.1. Elastic impact

Consider the arrangement shown in Figure 3 in which an iscosceles, right-angled triangle impacts the wall with uniform velocity $U_0\mathbf{i}$. After impact the wave face has horizontal velocity component U_2, which is the same at all points on the wave face. Hence the boundary condition at $x = 0$ is $\partial P/\partial x = \rho(U_0 - U_2)$. The solution, $P(x, y)$, is similar to the right-hand side of Equation (27):

$$P(x, y) = \rho \frac{(U_0 - U_2)}{2R} \left([x + R]^2 - y^2 \right). \tag{28}$$

Consequently the impulse $\mathbf{I} = I\mathbf{i}$ exerted by the wave on the block is given by

$$I = \int_0^R P(0, y)\mathrm{d}y = \frac{1}{3}\rho R^2(U_0 - U_2). \tag{29}$$

The block has a mass M, per unit length of shoreline, and moves from rest to a velocity $U_t\mathbf{i}$. The impulse on the mass equals its change of momentum: $I = MU_t$. Therefore from Equation (29)

$$\frac{1}{3}\rho R^2(U_0 - U_2) = MU_t. \tag{30}$$

For an elastic impact the relative speed of departure of the wave face and wall equals their relative speed of approach, hence

$$U_t - U_2 = U_0. \tag{31}$$

The solution conserves (kinetic) energy. The results are presented in terms of U_0 and the dimensionless ratio $q = \rho R^2/(3M)$. The block acquires a velocity $U_t = 2U_0 q/(1 + q)$. The total impulse on the block is $\mathbf{I} = 2\mathbf{i}MU_0 q/(1 + q)$. The horizontal velocity component of the wave face after impact $U_2 = -U_0(1 - q)/(1 + q)$. The field of pressure impulse, P, is given by replacing U_0 by $2U_0/(1 + q)$ in the expression (27)

If the mass M of the block is much greater than the mass $\rho R^2/2$ of the wave then $q \ll 1$ and we recover the results for elastic impact on a *rigid* wall. Under these circumstances the impulse $\mathbf{I}$ achieves its greatest magnitude. At the other extreme, if $M \ll \rho R^2/2$ then the limit $q \to \infty$ reveals the wave velocity is unchanged from U_0 and the block is kicked forwards with a velocity $2U_0\mathbf{i}$. If $q = 1$ then the front face of the wave is brought to a halt and the block moves on with speed $U_t = U_0$. The condition $q = 1$ means $M = \rho R^2/3$, which is two-thirds of the mass of the wave.

5.2. Inelastic impact

More realistic than the above treatment is the assumption that the face of the wave remains in contact with the block after impact, so that $U_2 = U_t$. The impulse I due to the fluid on the block is given by (29) and equals

$$\frac{1}{3}\rho R^2(U_0 - U_t) = MU_t, \tag{32}$$

which implies

$$U_t = U_0\frac{q}{1+q}, \tag{33}$$

where, as before, $q = \rho R^2/(3M)$. For inelastic impact U_t is half of the speed found earlier for elastic impact. The impulse on the wall is

$$I = MU_0\frac{q}{1+q}, \tag{34}$$

which is half the impulse for elastic impact.

Energy is not conserved. In the notation of Section 3 the kinetic energy of the system before impact lies wholly in the wave:

$$KE_1 = \frac{1}{4}\rho R^2 {U_0}^2 \tag{35}$$

and after impact the energy of the wave and the block are together given by

$$KE_2 = \rho R^2 \left(\frac{U_2{}^2}{6} + \frac{U_0{}^2}{12} \right) + \frac{1}{2} M U_t{}^2. \tag{36}$$

When the results above are used, the loss of KE simplifies to

$$KE_1 - KE_2 = \frac{1}{6(1+q)} \rho R^2 U_0{}^2, \tag{37}$$

which is strictly positive for all $q \geq 0$. Hence energy is always lost from the system. Most energy is lost when $q = 0$ i.e. when the block has much greater mass than the wave. If the block has negligible mass then $q \to \infty$ and the block gains no energy and hence the wave loses none.

5.3. Block displacement: inelastic impact

The displacement is useful to model because it can be compared with simple measurements of damaged harbour works, and give a hindcast of the pressures which the waves might have exerted.

For the motion of the block after impact we neglect the remaining force of the wave and suppose that the block is brought to rest by the friction between the horizontal base and the horizontal surface over which it slides. We calculate the total distance the block moves. If the coefficient of friction is a constant, μ, then the displacement d is given by (Cox [25])

$$d = \frac{U_t{}^2}{2\mu g} = \frac{U_0{}^2}{2\mu g} \frac{q^2}{(1+q)^2}. \tag{38}$$

The first of the above equations is obtained by equating the initial kinetic energy of the block with the work done in moving it a distance d against friction.

The expressions in (38) have several revealing features. The displacement, d is directly proportional to the *square* of the speed of impact (hence the square of the impulse I). The displacement is small when q is small, *i.e.*, when M is large compared with $\rho R^2/3$, which is two-thirds of the wave mass. The distance d increases from zero as q increases from zero, to a hypothetical maximum displacement of $U_0{}^2/(2\mu g)$. However, in this limit the block is so light-weight that friction is no longer the only significant force on the block: then the wave force after impact must also be taken into account. In practice for seawall blocks or caissons the range of practical interest is at the lower end of the interval for $q : 0 < q < 3$, approximately. Hence the maximum displacement might be much less than $U_0{}^2/(4\mu g)$.

As an example we take a solid block, of rectangular cross-section, of height equal to $R = 10$m, width 20m and density 2·5 times greater than the density of water. The coefficient of friction $\mu = 0{\cdot}6$ (Goda [24, Section 4.3]). A wave impacts the block with a speed $U_0 = \sqrt{gR} = 10$m/s. Then according to Equation (38) the block translates a distance of about 0·033m.

If the block has half the mass, because it has square cross-section (height=width=$R = 10$m) then, with the other conditions unchanged, the block moves much further: $d = 0{\cdot}12$m.

In practice such displacements might be lessened by the frictional resistance to motion due to any neighbouring blocks in a structure which are not being impacted by the wave. The displacement might be uphill or the block could pivot about its rear edge. Also we have neglected friction *during* impact; accounting for this would lessen the initial velocity of the block. However, there is ample evidence that wave-impacts do move sections of vertical breakwaters impressively large distances. Stevenson [26, 27] reports the destruction of the breakwater at Wick Harbour: a straight, monolithic mass of thousands of tonnes of concrete. This moved a distance at least as great as the width of its foundations by breaking waves.

Recently Hitachi [28] reports displacements of between a few centimetres and 5m for caissons 18m high and 24m wide, in a Japanese breakwater, due to high impacting waves during one storm in 1991. A calculation, using Equation (38), for a caisson with these dimensions, suggests a displacement of $d = 0{\cdot}12$m due to one impact. A succession of wave impacts, (a likely assumption during one storm) could therefore displace a caisson the distances measured by Hitachi [28].

6. Compressible fluid impact: isosceles triangle

So far we have modelled the fluid as incompressible. It has long been suspected that impact pressures are sufficiently high that the compressibility of water might be a significant factor in accounting for impact pressures. But only in the last decade have the effects begun to be quantified. The presence of even two percent by volume of air bubbles in sea water, generated by breaking waves, dramatically reduces the speed of sound from around 1500m/s to less than 100m/s; see Van Wijngaarden [29]. Here we assume that the wave is a homogeneous medium of well-mixed water and small air bubbles.

We model the wave water as compressible so that pressure disturbances travel as signals which have a constant speed of sound, c. Such impact problems have been solved by Frankel [30] and Korobkin [6]. The latter paper contains a detailed justification for using linear acoustics to model the kind of impact discussed below.

During impact the unsteady field of velocity $\mathbf{V}(x, y, t)$ can be described by a velocity potential $\phi(x, y, t)$, where $\nabla\phi = \mathbf{V}$. The associated deviation of pressure, from constant atmospheric pressure, is $p(x, y, t) = -\rho_0\phi_t$. The following model equations are used in acoustics and have already been linearised from Euler's equations under the assumptions of low Mach number ($|\mathbf{u}|/c \ll 1$) and negligible nonlinear terms. (The variables x, y are the usual cartesian coordinates.)

$$u_t = -\rho_0{}^{-1} p_x, \tag{39}$$

$$v_t = -\rho_0{}^{-1} p_y, \tag{40}$$

where ρ_0 is the equilibrium density.

Next is a linearised equation of mass-continuity, for the density ρ. The density is assumed (i) to be a function of pressure alone, and (ii) to differ little from its equilibrium value.

$$\rho_t = -\rho_0(u_x + v_y). \tag{41}$$

To close the model we have an effective equation of state:

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = c^2. \tag{42}$$

From Equations (39) to (42) it is easy to show that ϕ satisfies the linear wave equation

$$c^{-2}\phi_{tt} = \phi_{xx} + \phi_{yy}. \tag{43}$$

The equations admit discontinuities in u, v, p, ρ governed by suitable jump conditions.

In the following the wave is an idealised shape in order that we may calculate an exact solution. We take the same arrangement as shown in Figure 2 for a sea wave, idealised as an isosceles right-angled triangle, which meets an impermeable rigid wall at $x = 0$, with normal component of velocity U_0 at impact. For simplicity we take the velocity field of the wave, before impact, to be $\mathbf{V}_1 = U_0\mathbf{i}$. The wave face stays in contact with the wall throughout impact.

The domain of the initial-boundary-value problem for ϕ is augmented by anti-reflection of the triangle in its free surface, to form a square domain, S: $-R \leq x \leq 0$ and $0 \leq y \leq R$. The boundary conditions are $\partial\phi/\partial x = U_0 - U_0 H(t)$ on $x = 0$ and $\partial\phi/\partial y = U_0 H(t)$ on $y = R$, where $H(t)$ is Heaviside's function. The normal derivative of ϕ is zero on the other two sides of S. At $t = 0$ the initial conditions are $\phi = U_0 x$, and $\phi_t = 0$.

We solve by setting ϕ equal to the sum of a quadratic potential (to accommodate the boundary conditions) and a double-sum over harmonic modes. Each of these satisfies the homogeneous derivative boundary conditions and takes the following form:

$$A_{mn} \cos\left(\frac{m\pi x}{R}\right) \cos\left(\frac{n\pi y}{R}\right) \cos\left(\frac{\pi ct\sqrt{m^2+n^2}}{R}\right),$$

where m and $n \in \{0, 1, 2, 3 \cdots\}$ and the Fourier coefficients A_{mn} are determined by our initial data. It turns out that $A_{00} = 0$ and $A_{mn} = 0$ unless either $m = 0$ or $n = 0$. Consequently the double-sum in the general solution folds up into just one summation, over time-periodic terms:

$$\phi(x, y, t) = U_0 x - \frac{H(t)U_0}{2R}\left([x+R]^2 - y^2 - \frac{4R^2}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos\frac{n\pi ct}{R}}{n^2}\left[\cos\frac{n\pi x}{R} - (-1)^n \cos\frac{n\pi y}{R}\right]\right). \tag{44}$$

The associated pressure is

$$p(x, y, t) = H(t)\rho U_0 c \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi ct}{R}}{n}\left(\cos\frac{n\pi x}{R} - (-1)^n \cos\frac{n\pi y}{R}\right). \tag{45}$$

The summations in (44, 45) can be written in closed form, but it is easier to discuss the results graphically. Figure 4 illustrates the following discussion, which is a physical interpretation of the formal solution (44, 45).

At the start of impact ($t = 0$) a compression-wave front AB moves leftward from the wall. Immediately AB begins to reflect (from the free surface, $y = x + R$) another, horizontal wave-front BC, of equal and opposite amplitude. BC descends from W towards the bed. Both wave fronts travel at speed c, and their associated pressure fields superpose. Outside the rectangle ABCO, and below the free surface, the pressure is zero, and the arrangement portrayed in Figure 4 satisfies the zero-pressure boundary condition at the free surface. The wave fronts are equally inclined, at $\pi/4$ to the plane free-surface. Inside ABCO the pressure is time-invariant (for the moment) and equal to p_J, which is also spatially uniform in ABCO. From $t = 0$ to $t = R/c$, p_J is the so-called water-hammer pressure $\rho c U_0$. The time $t = R/c$ coincides with the instant at which AB reaches the tail T of the triangle, and equals the instant at which BC reaches the bed. The wave front reflects from the bed and T. After reflection BC rises and its reflection in the free surface, AB, propagates to the right. ABCO now contains

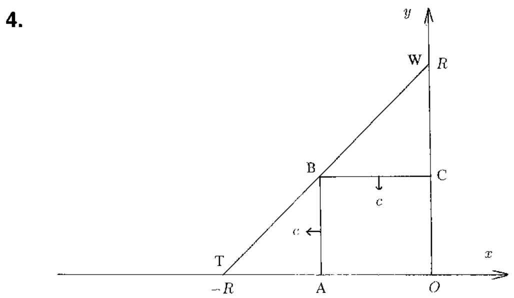

Figure 4. A compression wave front AB moves at the speed of sound, c, away from the wall. It reflects from the free surface, making a second wave front BC. In the region ABCO the pressure is uniform and equals the constant $\rho_0 U_0 c$. After BC reflects from the bed, the pressure in ABCO changes to $-\rho_0 U_0 c$. The compression waves repeatedly reflect from the wall, bed and free-surface, forming a standing wave system of period $2R/c$.

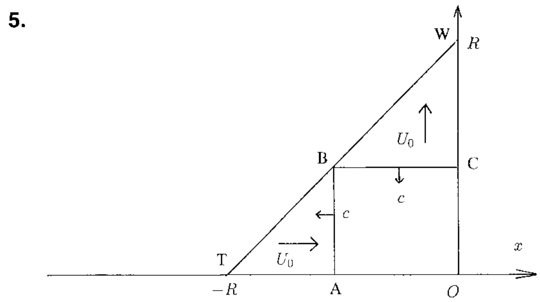

Figure 5. As Figure 4. The associated velocity fields are uniform in each of the three regions. Note that for a fluid particle near W the fluid ascends with velocity $U_0\mathbf{j}$ for a time much longer than for a particle near the bed. The time-averaged velocity varies linearly with position in accord with incompressible fluid theory.

fluid at uniform pressure $-\rho_0 U_0 c$. At time $t = 2R/c$ the wave front AB returns to the wall. Subsequent reflections repeat the cycle described for $t : 0 \leq t \leq 2R/c$. The wave-fronts describe a standing wave pattern for as many periods as we may expect the water wave domain to remain close to an isosceles triangle.

Next we find the changes in the velocity during impact. Figure 5 shows the velocity fields in the three regions of the triangle during impact. As AB moves left the velocity component normal to the wave front changes according to the following jump condition, obtained from integrating (39) across AB from the state '1', before impact, to state 'J', inside ABCO.

$$c[u]_1^J = -\rho_0^{-1}[p]_1^J. \tag{46}$$

Therefore, from Equation (45)

$$c\,(u_J(-ct, y) - U_0) = -\rho_0^{-1}\rho_0 U_0 c, \tag{47}$$

hence

$$u_J(x, y) = 0 \qquad \text{for} \quad (x, y) \in ABCO \tag{48}$$

and in ABCO the vertical velocity is unchanged from zero, its initial state.

As BC descends the wave front alters the vertical velocity component. Equation (40) gives the jump condition from state 'J' inside ABCO to a state '2' above front BC, in the region BWC.

$$c[v]_J^2 = -\rho_0{}^{-1}[p]_J^2. \tag{49}$$

From Equation (45)

$$cv_2(x, R - ct) = \rho_0{}^{-1}\rho_0 c U_0, \tag{50}$$

hence

$$v_2(x, y) = U_0 \qquad \text{for} \quad (x, y) \in BWC. \tag{51}$$

Also BC leaves unchanged the horizontal velocity component, so $u_2 = u_J = 0$ in BWC.

At time $t = R/c$ the wave front BC reflects from the bed. Inside ABCO the pressure changes to $-\rho_0 U_0 c$ and the velocity field reverts to $U_0\mathbf{i}$ in TAB, (and zero velocity inside ABCO) as AB and BC sweep back across the triangle.

The time-average of the velocity potential (44), over one period $t : T \leq t \leq T + 2R/c$, where $T > 0$ is arbitrary, is

$$\phi_m(x, y) = \frac{U_0}{2R}(R^2 - x^2 + y^2), \tag{52}$$

defined throughout the triangle. This agrees with the sudden change in velocity potential predicted by (27) from the theory for an incompressible fluid: $P = \rho_0(\phi_1 - \phi_2)$, where we take $\phi_1 = U_0 x$ and $\phi_2 = \phi_m$. The potential (52) can be used to calculate the fluid displacement over an integer number of compression wave periods. The nett displacement is non-zero and the fluid spreads up the wall.

The velocity potential (44) contains oscillatory time-dependent terms. In a dissipative medium the terms $\cos(n\pi ct/R)$ are supplemented by factors which decay exponentially with increasing t. So we can expect the potential to evolve to leave $\phi = U_0 x - U_0/2R([x + R]^2 - y^2) = \phi_m$, after a suitably large number of periods. Under these circumstances the time-integral of p from $t = 0$ to $t = \infty$ is close to the pressure impulse (27). The distortion of the free surface with increasing time disrupts the perfectly repeated reflections modelled here, hence the standing compression waves might degenerate into a complex web of reflections, whose average velocity potential might be identified with ϕ_m, as given by (52).

The standing pressure-wave field contains potential energy which is unable to radiate to infinity. This is in contrast with the radiation found in the unbounded geometries treated by Korobkin [6] and Korobkin and Peregrine [18]. Compression waves store within the triangle some of the energy that is lost from the kinetic energy of the incident wave. The remainder is accounted for by the constraint that the fluid remains in contact with the wall after impact.

7. Conclusions

By treating Lagrange's equations of motion we modify our interpretation of the pressure impulse to the time-integral of the fluid pressure following the motion of each fluid particle. The classical free-surface boundary condition $P = 0$ is confirmed. It is appropriate to solve for P in a domain which coincides with the physical fluid domain at the instant of impact.

During the impact of an incompressible fluid there is a well-known loss of (kinetic) energy. This is known to be directly attributable to the inelastic boundary condition usually applied on the basis of common experience. But a condition of elastic rebound is the only impact-boundary condition which admits kinetic energy conservation for an incompressible fluid.

Some simple potentials are suitable solutions for P in models of fluid impact. For the impact of an acute-angled wedge the solution is linear, and therefore predicts a non-singular, post-impact velocity field. However, for wedge angles which are obtuse the solution has a singular derivative at the vertex, corresponding to unbounded fluid speed after impact.

An exact solution is presented for impact on the interior of a circular cylinder. It is shown that the confines of the cylinder enhance the impulse on the bed of the container. Another solution, suitable for a triangle of fluid impacting a rigid wall illustrates some matters of sea wave impact on a vertical wall. The solution is adapted in Section 5 to wave interaction with a movable wall. The well known energy loss is not explained away by allowing the wave to do work on a movable wall. The sliding of the block after impact is small if the mass of the block is much greater than the mass of the wave.

For the same triangular wave, but with a compressible fluid, we present an exact solution to the linear acoustic equations. The solution contains standing waves which store some of the energy lost from the kinetic energy of the incident water wave. The solution implies that the time-averaged fluid velocity is the same as that predicted by pressure impulse theory for an incompressible liquid.

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